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Elastic modulus imaging: on the uniqueness and nonuniqueness of the elastography inverse problem in two dimensions

Paul E Barbone and Nachiket H Gokhale

Aerospace and Mechanical Engineering, Boston University, 110 Cummington Street, Boston, MA 02215, USA

E-mail: barbone@bu.edu

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Abstract
We examine the uniqueness of an N-field generalization of a 2D inverse problem associated with elastic modulus imaging: *given N linearly independent displacement fields in an incompressible elastic material, determine the shear modulus*. We show that for the standard case, $N = 1$, the general solution contains two arbitrary functions which must be prescribed to make the solution unique. In practice, the data required to evaluate the necessary functions are impossible to obtain. For $N = 2$, on the other hand, the general solution contains at most four arbitrary constants, and so very few data are required to find the unique solution. For $N = 4$, the general solution contains only one arbitrary constant. Our results apply to both quasistatic and dynamic deformations.

1. Introduction

Elastography, the imaging of soft tissue on the basis of (shear) elastic modulus, is an emerging imaging method. The motivation underlying elastography stems from the fact that abnormal tissue is often firmer (stiffer) than normal tissue. One example of this can be found in breast cancer: about half of all breast cancers detected in the USA are first discovered by the patient feeling a hard lump in her breast [1]. Another example is arteriosclerosis, which literally translates from Greek to 'hardening of the arteries' [2]. Still other examples can be found in fibrosis, deep vein thrombosis, liver, prostate and other cancers, and in treatment monitoring applications. These and other applications have recently received the attention of researchers trying to devise techniques to measure tissue stiffness *in vivo*[1].

[1] The special issue [3] covers many applications and techniques. For representative efforts in breast and other cancers, for example, see [4–9]. For applications to intravascular imaging, see [10–16]. Deep vein thrombosis is described in [17], while feasibility studies of monitoring thermal ablation treatments are presented in [18–21].

All published techniques rely on being able to image soft tissue while it is being deformed by a set of externally applied forces. Through image processing, the displacement (or sometimes velocity) field everywhere in the region of interest is inferred. An inverse problem for the elastic modulus is formulated, given the measured displacement fields, an assumed form of the tissue's constitutive equation (e.g. linear elastic), and the law of conservation of momentum. In some cases, the relevant inverse problem is associated with incompressible elastostatics (e.g. [8, 9, 22–24]), while in others, incompressible elastodynamics is more relevant (e.g. [7, 25–31]). In this paper, we study the uniqueness of the linear elastic form of this inverse problem.

This inverse problem has some unique features which stem from the way data are acquired. Medical imaging equipment permits the measurement of displacement fields *throughout* a region of interest of a body. Thus we are not limited to data taken on the *boundary* of the body, as is typical in many inverse problems. In this respect, it is analogous to an inverse problem in coastal evolution [32–34]. This results in an inverse problem which violates many of the 'rules' of inverse problems to which some readers may be accustomed. For example, as we see below, the operator to be inverted for the sought parameter is apparently noncompact, and therefore the role of regularization may be reasonably called into question.

The inverse problem considered is essentially this: given the displacement field $\boldsymbol{u}(x, y)$ (or in some cases $\boldsymbol{u}(x, y, t)$) in an incompressible linear elastic solid throughout a (2D) region of interest Ω, determine the shear modulus $\mu(x, y)$ in Ω. We note again that we are given the displacement fields throughout a nontrivial region of interest within the object. This leads to a second order linear partial differential equation (pde) with variable coefficients for the sought shear modulus (cf equation (6) below). The boundary data required to render the solution of this pde unique are typically unobtainable in practice.

This ambiguity was first recognized by Skovoroda and co-workers [8]. They derived equation (6) in its quasistatic form and noted that it was hyperbolic. They described Goursat data to render the solution unique and considered a means to determine closed contours on which $\mu =$ constant to serve as boundaries within which the modulus could be determined uniquely. Unfortunately, their technique to find such contours is unreliable, in the sense that it is possible to find examples for which it fails [35]. Furthermore, as we demonstrate in an example below, even if it were possible to find such contours reliably, their use with the pde (6) leads to an ill-posed boundary value problem. Barbone and Bamber [35] consider at length the question of uniqueness given a single displacement field. They show among other things that in certain circumstances, sufficient knowledge of the boundary conditions in the *forward* problem can provide enough information to render the inverse problem unique. Those circumstances, however, are practically impossible to realize in medical imaging practice.

Very recently, McLaughlin and Yoon investigated the role of such boundary conditions in determining the uniqueness of the *compressible* form of the elastography inverse problem [36]. We note that the compressible form of the elastic inverse problem is fundamentally different from the incompressible form. We believe the incompressible approximation is more appropriate to the problem of soft tissue imaging, and so we discuss that case in detail here. Essentially, our preference is based on the recognition that the nearly incompressible inverse problem is highly sensitive to small errors in the measured dilitation ($\nabla \cdot u$), while the incompressible case is insensitive to these errors.

In order to devise a method by which the shear modulus could be determined reliably, therefore, some means is required to increase the information available to us beyond a single measured displacement field. A logical extension is to consider measuring two (or more) different displacement fields that would result from two (or more) different kinds of external excitation. Indeed, we show below that, given multiple displacement fields, very little additional information is required in order to reconstruct the shear modulus uniquely. In particular, with just two displacement fields, the shear modulus distribution is determined uniquely with four or fewer *a priori* known values of μ. With four different displacement fields, the shear modulus is determined uniquely up to a multiplicative constant with no additional information regarding μ.

Below, we first discuss the forward problem formulation, and then the corresponding inverse problem. We derive the linear second order pde with variable coefficients that is satisfied by μ. We then demonstrate the nonuniqueness with a single displacement field through several examples. To the student of elastic inverse problems, the examples are intended to provide some insight into the character of the problem. To the critic, the solutions provide counter-examples to the assertion 'it is possible to uniquely determine the shear modulus from a single measured displacement field'. The examples include special cases in 2D of both quasistatic and dynamic deformations. We note that since plane strain (our 2D approximation) represents a special case of 3D, these solutions also provide counter-examples to the same assertion made regarding the 3D inverse problem. That is, the examples demonstrate the theoretical possibility of nonuniqueness in 3D as well as in 2D. We also demonstrate the ill-posedness (in the sense of Hadamard; cf [37]) of the inverse problem with known stiffness on the boundaries of the domain.

The bulk of this paper, however, is dedicated to the discussion of uniqueness of the inverse problem in 2D when multiple displacement fields are available. We show that in contrast to the single-field case, using two displacement fields simultaneously gives rise to a solution space that is at most four dimensional. Thus, finding the unique solution with two displacement fields requires *a priori* knowledge of the stiffness at only four points. With four distinct displacement fields, the stiffness is determined uniquely up to a multiplicative constant. We give the exact solution for this case in terms of quadratures.

2. Formulation: forward incompressible elasticity problem

We consider a 2D body occupying the region Ω, with boundary Γ. We let $\boldsymbol{u}(x, y) \equiv \boldsymbol{u}(x_1, x_2) \equiv \boldsymbol{u}(\boldsymbol{x})$ denote the displacement field as a function of the spatial coordinate $\boldsymbol{x}$. The linearized strain components, measured with respect to Cartesian axes, are given by

$$\epsilon_{ij} = \tfrac{1}{2}[\partial_i u_j + \partial_j u_i]. \tag{1}$$

In (1) we have introduced the shorthand notation $\partial_j = \frac{\partial}{\partial x_j}$.

The incompressibility constraint can be expressed as

$$\sum_{k=1}^{2} \epsilon_{kk} \equiv \epsilon_{kk} = \partial_k u_k = 0. \tag{2}$$

In writing (2), we introduce the summation convention over repeated indices which we will continue to use throughout the paper. We also note that equation (2) implies that we are considering the plane strain approximation of 2D elasticity.

Finally the stress–strain relation, or constitutive equation, for an incompressible, linear elastic material, is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\epsilon_{ij}. \tag{3}$$

In (3), σ_{ij} is a component of the stress tensor, σ; p is the pressure or the hydrostatic part of the stress; μ is the material shear modulus, which in some texts is denoted by G. It is related to Young's modulus E and Poisson's ratio ν through $\mu = G = E/2(1 + \nu)$ [38]. For an incompressible material, $\nu = 1/2$.

The balance of linear momentum over each part of the material is given by

$$\partial_j \sigma_{ij} = \rho \partial_{tt} u_i. \tag{4}$$

In writing (4) we have neglected all body forces. Equation (3) can be substituted into (4) to write the momentum equation directly in terms of the pressure and the displacement:

$$-\partial_i p + \partial_j(\mu \partial_i u_j) + \partial_j(\mu \partial_j u_i) = \rho \partial_{tt} u_i. \tag{5}$$

Equations (5) and (2) are sufficient to define the pressure and displacement fields everywhere in the solid, when appropriate boundary conditions are specified.

2.1. Forward problem statement

The forward problem may now be stated thus: *given suitable boundary conditions defined on* Γ, *and* μ *and* ρ *defined on* Ω, *find* $\boldsymbol{u}$ *and* p *such that equation* (5) *is satisfied.*

2.2. Inverse problem statement

The static (respectively dynamic) inverse problem with which we are concerned is the following: *given* $\boldsymbol{u}(\boldsymbol{x})$ $(\boldsymbol{u}(\boldsymbol{x}, t)$ *and* $\rho =$ constant) *defined on* Ω $(\Omega \times (0, T > 0))$, *find* $\mu(\boldsymbol{x})$ *and* $p(\boldsymbol{x})$ *such that equation* (5) *is satisfied.*

We have obviously neglected to mention any boundary data in this problem statement. In the following, we will consider various situations and will make it clear what the given boundary data are in each case. Generally speaking, the problem as given here is ill posed without additional data.

Equation (5) gives us directly a set of pdes that we can (potentially) solve for μ and p. The easiest way to analyse the system for μ is to eliminate pressure, p. To that end, we take the curl of (5) and obtain for the z-component

$$(\partial_{yy} - \partial_{xx})(\epsilon_{xy}\mu) + 2\partial_{xy}(\epsilon_{xx}\mu) = \tfrac{1}{2}\rho \partial_{tt}(\partial_y u_x - \partial_x u_y). \tag{6}$$

In simplifying (6), we used the incompressibility condition (2). The right-hand side of (6) represents the inertia term, which would be identically zero in the case of quasistatic deformation. We note that since ρ and $\boldsymbol{u}$ are known, the inertia term on the right-hand side of (6) is known.

3. Examples of nonuniqueness

In this section we present several examples which demonstrate the nonuniqueness inherent in the inverse problem associated with elastic modulus imaging. We discuss examples for both quasistatic and transient displacement fields. We remark here that we are working in the plane strain assumption, which is an exact solution of the 3D equations. Therefore, the examples of nonuniqueness given here can also serve as examples of nonuniqueness of the 3D problem.

3.1. Nonuniqueness from a quasistatic deformation

Consider the equilibrium displacement field given by

$$u_x = \epsilon x \tag{7}$$

$$u_y = -\epsilon y. \tag{8}$$

This is the displacement that would result if a homogeneous material were compressed uniformly in the y-direction, with an overall strain of $-\epsilon$. It is natural to ask whether the inverse problem has $\mu =$ constant as its only solution.

In this case, equation (6) simplifies to

$$\partial_{xy}\mu = 0. \tag{9}$$

That is, the general solution to this inverse problem is

$$\mu(x, y) = f(x) + g(y), \tag{10}$$

in which $f(x)$ and $g(y)$ are arbitrary functions of their arguments. Clearly, the solution space is doubly infinite dimensional.

3.2. Nonuniqueness from a transient deformation

Here we shall use the commonly employed assumption that ρ = constant and is known. We now consider the displacement field:

$$u_x = 0 \tag{11}$$

$$u_y = A\sin(kx)\cos(\omega t). \tag{12}$$

Substituting into (6) shows that $\mu(x, y)$ must satisfy the following equation:

$$\partial_{yy}\mu - \partial_{xx}\mu - 2k\tan(kx)\partial_x\mu + k^2\mu = \rho\omega^2. \tag{13}$$

Equation (13) is an inhomogeneous second order linear hyperbolic pde with nonconstant coefficients. Two different solutions of (13) are

$$\mu_1(x, y) = \frac{\rho\omega^2}{k^2} \tag{14}$$

$$\mu_2(x, y) = \frac{\rho\omega^2}{k^2}(1 + A\cos(ky)). \tag{15}$$

In (15), A is an arbitrary constant. Thus the given displacement field is consistent with more than one modulus distribution.

More generally, the general solution of (13) can be written as

$$\mu(x, y) = \frac{\rho\omega^2}{k^2} + \mu^{\mathrm{h}}(x, y), \tag{16}$$

in which $\mu^{\mathrm{h}}(x, y)$ is any solution of the homogeneous equation

$$\partial_{yy}\mu^{\mathrm{h}} - \partial_{xx}\mu^{\mathrm{h}} - 2k\tan(kx)\partial_x\mu^{\mathrm{h}} + k^2\mu^{\mathrm{h}} = 0. \tag{17}$$

While we cannot construct the closed form general solution of (17), we know from the Cauchy–Kowalewski theorem [37] that the number of arbitrary functions in the general (analytic) solution is exactly two, just as in the quasistatic example of equation (10).

In passing, we note that equation (13) is singular along the line $kx = n\pi - \pi/2$. An examination of (12) and the definition of strain (1) shows that along these lines the given strain field is zero.

3.3. Example of unique but ill-posed inverse problem

In this example, we assume that we know that the shear modulus $\mu(x, y)$ is constant on the boundary of our domain. This is consistent with the approaches taken in [8, 24, 23]. For definiteness, we consider the domain $\Omega = (0, 1) \times (0, h)$, and suppose that μ satisfies the boundary conditions:

$$\mu(x, 0) = \mu(x, h) = \mu(0, y) = \mu(1, y) = \mu_0. \tag{18}$$

Here, h is the nondimensional height-to-width ratio.

Now we consider the equilibrium displacement field given by

$$u_x = \epsilon y \tag{19}$$

$$u_y = 0. \tag{20}$$

This is the displacement that would result if a homogeneous material were subjected to uniform shear strain.

In this case, equation (6) simplifies to

$$(\partial_{yy} - \partial_{xx})\mu = 0. \tag{21}$$

Certainly, a solution of (21) that satisfies the boundary conditions (18) is

$$\mu(x, y) = \mu_0 + \mu^{\mathrm{h}}(x, y). \tag{22}$$

The function $\mu^{\mathrm{h}}(x, y)$ must satisfy (21) and (18), but with μ_0 replaced by 0. This problem is unique provided that the only solution for μ^{h} is $\mu^{\mathrm{h}} = 0$. Otherwise, it is not unique.

Consider the function

$$\mu^{\mathrm{h}}(x, y) = \sin(n\pi x)\sin(m\pi y). \tag{23}$$

Equation (23) satisfies (21) provided that

$$(n^2 - m^2) = 0. \tag{24}$$

Furthermore, equation (23) satisfies the homogeneous form of (18) provided

$$n = \text{integer} \tag{25}$$

$$mh = \text{integer}. \tag{26}$$

Clearly, if $h = N/M$ is rational, then, $n = m = M$ satisfies all conditions (24)–(26). If, on the other hand, h is irrational, then it is impossible to find an integer m that will satisfy (26).

In fact, in the case where h is irrational, the solution $\mu = \mu_0$ is the unique solution of (19) and (18). For any irrational h, however, one may choose a rational number, $\bar{h} \approx h$, that approximates h as accurately as one would like. For example, if one wanted to approximate h to 50 decimal places, one could simply choose $\bar{h}$ to be the first 50 decimal places of h. The corresponding boundary value problem with height $\bar{h}$ instead of h is obviously nonunique, since by definition $\bar{h}$ is rational.

The argument above shows that the solution of (19) with boundary condition (18) is not a continuous function of its parameters (in particular, h). It is, therefore, ill posed in the sense of Hadamard (cf [37]). In general, Dirichlet boundary conditions (such as those prescribed in (18)) are inappropriate for hyperbolic pdes such as (6).

4. Uniqueness with multiple displacement fields

The examples of the previous section demonstrate the form of the nonuniqueness inherent in the elastic modulus imaging inverse problem. The basic issue is that the shear modulus μ satisfies a second order hyperbolic pde. Without knowledge of boundary conditions appropriate for the equation, the solution cannot be determined uniquely.

In this section, we reconsider the elastic modulus imaging inverse problem when we are given two or more independent displacement fields. We show that the unique determination of the shear modulus requires very little *a priori* knowledge about the modulus itself, even when only two displacement fields are known.

The revised statement of the inverse problem may be given thus.

4.1. Multiple-displacement-field inverse problem

Given N displacement fields $\boldsymbol{u}^{(n)}(\boldsymbol{x}, t)$ *on* Ω, *find* $\mu(\boldsymbol{x})$ and $p^{(n)}(\boldsymbol{x}, t)$ $n = 1, \ldots, N$, *so that* (5) *is satisfied for each case. That is*

$$-\partial_i p^{(n)} + \partial_j(\mu\partial_i u_j^{(n)}) + \partial_j(\mu\partial_j u_i^{(n)}) = \rho\partial_{tt}u_i^{(n)} \qquad n = 1, \ldots, N. \tag{27}$$

We emphasize the crucial point that the function μ that appears in (27) is the *same* function for each value of n.

4.2. Two displacement fields

Here we suppose we are given two distinct displacement fields; $N = 2$. We shall show that in this case the set of possible solutions of the inverse problem is at most four dimensional. This implies that only four μ data points are required to be known *a priori* in order to determine μ uniquely. We will reach this result, the main result of the paper, through a series of lemmas. The first is related to a general property of the inversion equation (6).

Lemma 1. *Given* $\boldsymbol{u}(\boldsymbol{x})$ *and* ϵ *computed from equation (1) and satisfying equation (2). If* $\epsilon \neq 0$ *and regular everywhere inside* Ω, *then we have the following.*

(i) Equation (6) is hyperbolic.
(ii) The characteristics of (6) are parallel to the principal axes of strain.
(iii) The characteristics of (6) are everywhere regular.

Proof. The discriminant for equation (6) is independent of the inhomogeneous term (the right-hand side) and is

$$d = 4\epsilon_{xx}^2 + \epsilon_{xy}^2. \tag{28}$$

Since $\epsilon_{xx} = -\epsilon_{yy}$ by equation (2), d is positive for any $\boldsymbol{\epsilon} \neq 0$. That the characteristics are parallel to the principal axes of strain may be verified by direct calculation. Finally, the regularity of the characteristics follows directly from the assumed regularity of $\boldsymbol{\epsilon}$ and the observation that through every point in the domain there are exactly two characteristic curves. To see this last point, consider the opposite possibility, that some point, say $\boldsymbol{x}_0$, has more than two characteristics through it. This implies that $\boldsymbol{\epsilon}$ has more than two eigenvectors at $\boldsymbol{x}_0$, which in turn implies that $\boldsymbol{\epsilon} = \epsilon_0\mathbf{1}$ for some constant ϵ_0, where $\mathbf{1}$ is the two-dimensional identity tensor. However, the incompressibility condition requires $\mathrm{tr}\,\boldsymbol{\epsilon} = 0$, and therefore $\epsilon_0 = 0$. On the other hand, the existence of at least two characteristics at each point (i.e. at least two eigenvectors) is guaranteed by the fact that $\boldsymbol{\epsilon}$ is real and symmetric. □

The next lemma shows how, from a limited number of data, the information contained in a pair of strain fields can be used to determine the shear modulus everywhere that both fields are known.

We note that the strain fields are assumed to be nonzero; it is impossible to determine the modulus in regions which are not deformed.

Definition 1. *Two displacement fields* $\boldsymbol{u}_1$ *and* $\boldsymbol{u}_2$ *(and their corresponding strain fields* $\boldsymbol{\epsilon}_1$ *and* $\boldsymbol{\epsilon}_2$*) are said to be 'compatible' with each other if they can each occur in the same material, that is, if there exists a function* μ *(not necessarily unique) that satisfies equation (6) for both* $\boldsymbol{\epsilon}_1$ *and* $\boldsymbol{\epsilon}_2$.

Lemma 2. *We suppose we are given two linearly independent compatible displacement fields, ϵ_1 and ϵ_2, that are everywhere nonzero in Ω. We shall further assume that on a curve $C = (x(s), y(s))$ we are given $\bar{\mu}(s) = \mu(x(s), y(s))$ and $\bar{\mu}_n(s) = \partial_n \mu(x(s), y(s))$, in which $\boldsymbol{n}$ is the unit vector normal to the curve. (That is, we are given Cauchy data for the hyperbolic pde.) Then, assuming the solution $\mu(x, y)$ exists (see lemma 3 below for a constraint on the Cauchy data necessary for existence), we have the following.*

(i) If the eigendirections of ϵ_1 are distinct from those of ϵ_2 except at isolated points, then μ is uniquely determined in all of Ω.
(ii) If there is a set of curves B_j $j = 1, \ldots, Q$, such that each B_j is a characteristic of both ϵ_1 and ϵ_2, and C intersects none of the B_j, then μ is determined uniquely in the smallest region bounded by $\cup(\Gamma, B_j)$ which contains C.

Proof. We rewrite (6) for each displacement field in symbolic form:

$$\mathcal{L}(\boldsymbol{u}_1)\mu = f(\boldsymbol{u}_1) \tag{29}$$

$$\mathcal{L}(\boldsymbol{u}_2)\mu = f(\boldsymbol{u}_2). \tag{30}$$

Without loss of generality, we shall assume the x and y axes are aligned with the characteristics of (29). Then, the Cauchy data on C and (29) are sufficient to determine μ uniquely in the rectangular shaded region D_1 shown in figure 1. D_1 is bounded by the characteristics of (29) that circumscribe C. (Remark: if C is a characteristic of (29), then interchange the roles of displacement fields $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.) We may now treat μ as known up the boundary of D_1, and regard μ and its derivatives on ∂D_1 as Cauchy data for equation (30). This determines μ uniquely in the quadrilateral D_2, which is bounded by the characteristics of (30) that circumscribe D_1 (see figure 1).

In this way, we see that equations (29) and (30) used alternately can define μ uniquely in increasingly larger and larger quadrilaterals. The growth of the quadrilaterals in any given direction depends upon the characteristics of (29) and (30) being nonparallel. Thus a curve B which is a characteristic of *both* equations (29) and (30) forms a limiting curve which is approached by one side of the quadrilateral D_j as $j \to \infty$. □

We note that in lemma 2 no limit was placed on the size of the initial curve C. That implies that the C may even be considered infinitesimal, and that in addition to the two equations (29) and (30) very few data indeed are required in order to determine μ. This point is made precise in the next lemma.

Lemma 3. *The Cauchy data specified on C satisfy a fourth order ordinary differential equation.*

Proof. For this proof, we will work in a coordinate system aligned with C. Therefore, we assume that C lies on the x-axis. The curve C may be (but need not be) a characteristic of either equation (29) or (30), but not both. The Cauchy data we are given are

$$\mu(x, 0) = \bar{\mu}(x) \tag{31}$$

$$\partial_y \mu(x, 0) = \bar{\mu}_n(x). \tag{32}$$

Equations (29) and (30) have a special form of which we shall take advantage. To see that, we rewrite equations (29) and (30) as

$$\mathcal{L}(\boldsymbol{u}_1)\mu = a^{(1)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(1)}\partial_{xy}\mu + c^{(1)}\partial_x\mu + d^{(1)}\partial_y\mu + e^{(1)}\mu = f^{(1)} \tag{33}$$

$$\mathcal{L}(\boldsymbol{u}_2)\mu = a^{(2)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(2)}\partial_{xy}\mu + c^{(2)}\partial_x\mu + d^{(2)}\partial_y\mu + e^{(2)}\mu = f^{(2)}. \tag{34}$$

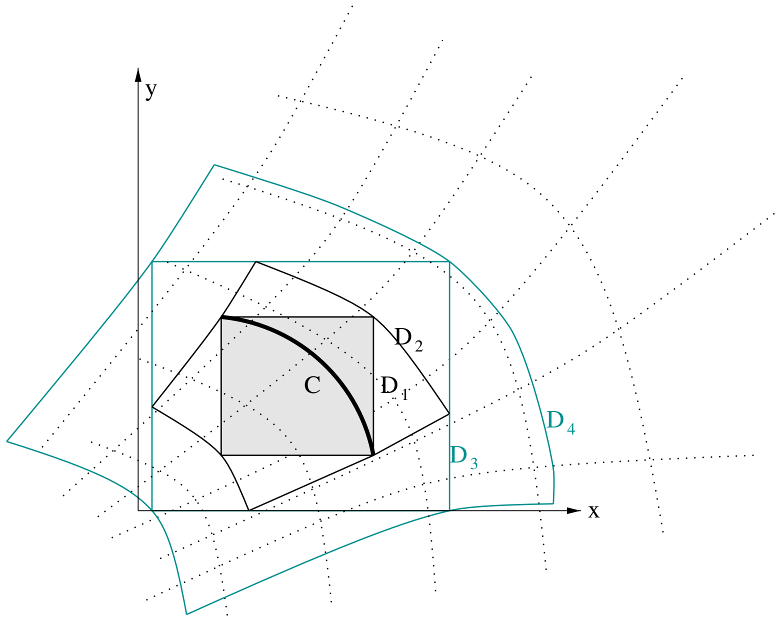

Figure 1. Filling the plane from limited initial data. Given Cauchy data on curve C, equation (29) can provide the solution in D_1, the shaded rectangle in the figure. From the knowledge of μ and its derivatives in D_1, equation (30) can provide the solution in the quadrilateral D_2. In the figure, the characteristic curves of equation (29) are aligned with the x and y axes, while the characteristic curves of equation (30) are shown dashed.

(This figure is in colour only in the electronic version)

In (33) and (34), the $a^{(j)}, b^{(j)}, \ldots, f^{(j)}$ are coefficients that are known in terms of the displacement components and their derivatives. In the calculations that follow, we shall continue introducing new coefficients $a^{(j)}, b^{(j)}, \ldots, f^{(j)}$, to simplify the forms of the resulting equations.

We now eliminate the y-derivatives in (33) and (34) in favour of the x-derivatives. To do so, we first take $a^{(1)} \times$ (34) $- a^{(2)} \times$ (33) to obtain

$$b^{(3)}\partial_{xy}\mu + c^{(3)}\partial_x\mu + d^{(3)}\partial_y\mu + e^{(3)}\mu = f^{(3)}. \qquad (35)$$

Similarly, we evaluate $b^{(3)} \times$ (33) $- b^{(1)} \times$ (35) to get

$$a^{(4)}(\partial_{yy} - \partial_{xx})\mu + c^{(4)}\partial_x\mu + d^{(4)}\partial_y\mu + e^{(4)}\mu = f^{(4)}. \qquad (36)$$

Proceeding, we compute $b^{(3)}\partial_x$ (36) $- a^{(4)}\partial_y$ (35) to find

$$a^{(5)}\partial_{xxx}\mu + b^{(5)}\partial_{xx}\mu + c^{(5)}\partial_x\mu + d^{(5)}\partial_y\mu + e^{(5)}\mu = f^{(5)}. \qquad (37)$$

In simplifying (37), we used (35) and (36) to eliminate any occurrences of $\partial_{xy}\mu$ and $\partial_{yy}\mu$, respectively. Next we evaluate $d^{(5)} \times$ (35) $- b^{(3)} \times \partial_x$ (37). This yields

$$a^{(6)}\partial_{xxxx}\mu + b^{(6)}\partial_{xxx}\mu + c^{(6)}\partial_{xx}\mu + d^{(6)}\partial_x\mu + e^{(6)}\partial_y\mu + f^{(6)}\mu = g^{(6)}. \qquad (38)$$

Finally, we eliminate $\partial_y\mu$ from (37) and (38) by evaluating $e^{(6)} \times$ (37) $- d^{(5)} \times$ (38) to obtain

$$a^{(7)}\partial_{xxxx}\mu + b^{(7)}\partial_{xxx}\mu + c^{(7)}\partial_{xx}\mu + d^{(7)}\partial_x\mu + e^{(7)}\mu = f^{(7)}. \qquad (39)$$

Equation (39) must be satisfied by any $\mu(x, y)$ which simultaneously satisfies equations (33) and (34). In particular, evaluating (39) on $y = 0$ and using equation (31) shows that $\bar{\mu}(x)$ must satisfy (39). Furthermore, equation (38), or equivalently (37), shows that $\bar{\mu}_n(x) = \partial_y \mu|_{y=0}$ is determined entirely in terms of $\bar{\mu}(x) = \mu(x, 0)$. □

Lemma 3 shows that the Cauchy data for the pair of equations cannot be imposed arbitrarily, but must satisfy some strict constraints. In particular, $\bar{\mu}(x)$ must satisfy (39), and once $\bar{\mu}(x)$ is chosen, $\bar{\mu}_n(x)$ follows from either (37) or (38). So, unlike the single-equation case, in which we have practically complete freedom to specify any Cauchy data we wish, the two-equation case is much more restricted. This observation leads us into the following theorem, which we consider to be the main result of this paper.

Theorem 1. *Given two linear independent compatible displacement fields, $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, that are everywhere nonzero, and such that the eigendirections of $\boldsymbol{\epsilon}_1$ are distinct from those of $\boldsymbol{\epsilon}_2$ except at isolated points, let $M^{(j)}$ be the set of all functions μ such that*

$$\mathcal{L}(\boldsymbol{u}^{(j)})\mu = 0. \tag{40}$$

Then,

$$\mathrm{Dim}\{M^{(1)} \cap M^{(2)}\} \leqslant 4. \tag{41}$$

In equation (41), Dim *stands for the dimension.*

Proof. Let $\bar{\mu}$ be a set of Cauchy data for one of the two equations, prescribed along the x-axis. Then by lemma 3, $\bar{\mu}$ satisfies a fourth order ordinary differential equation. Therefore, $\bar{\mu}$ can contain at most four arbitrary constants. By lemma 2 (i), $\bar{\mu}$ determines μ in all of Ω. Therefore, $\mu(x, y)$ contains at most four arbitrary constants. □

4.3. Example with two quasistatic strain fields

Here we consider a body which has been subjected to two different quasistatic deformation experiments. In the first experiment, the following strain field is measured:

$$\epsilon_{xx}^{(1)} = \epsilon_1 \tag{42}$$

$$\epsilon_{yy}^{(1)} = -\epsilon_1 \tag{43}$$

$$\epsilon_{xy}^{(1)} = 0. \tag{44}$$

When the body is subjected to a second, different, external force, a second strain field is supposed to be measured:

$$\epsilon_{xx}^{(2)} = 0 \tag{45}$$

$$\epsilon_{yy}^{(2)} = 0 \tag{46}$$

$$\epsilon_{xy}^{(2)} = \epsilon_2. \tag{47}$$

In the above equations, ϵ_1 and ϵ_2 are constants. We note that the two strain fields have no eigenvectors in common.

Substituting the strain field (42)–(44) into equation (6) shows that $\mu(x, y)$ satisfies and is given by (cf equation (10))

$$\partial_{xy}\mu = 0, \tag{48}$$

$$\mu(x, y) = f(x) + g(y). \tag{49}$$

Here the functions $f(x)$ and $g(y)$ are, up to now, arbitrary.

On the other hand, substituting (45)–(47) into (6) shows that $\mu(x, y)$ must *also* satisfy (cf equation (10)):

$$(\partial_{yy} - \partial_{xx})\mu = 0. \tag{50}$$

Substituting (49) into (50) gives

$$f''(x) = g''(y) = \text{constant} = 2C_4. \quad (51)$$

Integrating (51) to find $f(x)$ and $g(y)$ shows that the most general solution of *both* equations (48) and (50) is

$$\mu(x, y) = C_1 + C_2 x + C_3 y + C_4(x^2 + y^2). \quad (52)$$

The constants C_1, C_2, C_3 and C_4 can be evaluated if $\mu(x, y)$ is known at four points, or if $\mu(x, y)$ and three of its derivatives (e.g. μ, $\partial_x\mu$, $\partial_y\mu$ and $\partial_{xx}\mu$) are prescribed at one point. We note that not every set of derivatives is sufficient or appropriate to evaluate the coefficients. So, for example, one could not specify a nonzero value for $\partial_{xy}\mu$ at some point, because that would contradict equation (48).

4.4. More than two strain fields

We have seen that in order to determine the shear modulus distribution with a single displacement field, a large amount of *a priori* knowledge about the modulus itself is required in order to provide an adequate set of boundary conditions to solve (6) for $\mu(x, y)$ in the entire domain Ω. With two measured displacement fields, however, we require only four pieces of information regarding $\mu(x, y)$ (for example, *a priori* knowledge of μ at four distinct points in Ω) to determine μ everywhere inside Ω. The next logical step is to consider the possible utility of using more than two displacement fields. As we will see, four displacement fields determine μ uniquely, and it is impossible to have five independent displacement fields.

Theorem 2. *Given four linear independent mutually compatible displacement fields,* $\boldsymbol{u}^{(j)}$, $(j = 1, 2, 3, 4)$*, then the solution* $\mu(x, y)$ *of the set of equations*

$$\mathcal{L}(\boldsymbol{u}^{(j)})\mu = f^j \qquad j = 1, 2, 3, 4 \quad (53)$$

is unique up to a multiplicative constant. Further, the solution is given by

$$\mu(x, y) = \mu(x_0, y_0)\exp\left\{\int_{y_0}^{y}\int_{x_0}^{x}\tfrac{1}{2}(R_x + Q_y)\,\mathrm{d}x\,\mathrm{d}y - \int_{y_0}^{y} R\,\mathrm{d}y - \int_{x_0}^{x} Q\,\mathrm{d}x\right\}. \quad (54)$$

Proof. We begin by rewriting equations (53) in the form

$$\mathcal{L}(\boldsymbol{u}_1)\mu = a^{(1)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(1)}\partial_{xy}\mu + c^{(1)}\partial_x\mu + d^{(1)}\partial_y\mu + e^{(1)}\mu = f^{(1)} \quad (55)$$

$$\mathcal{L}(\boldsymbol{u}_2)\mu = a^{(2)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(2)}\partial_{xy}\mu + c^{(2)}\partial_x\mu + d^{(2)}\partial_y\mu + e^{(2)}\mu = f^{(2)} \quad (56)$$

$$\mathcal{L}(\boldsymbol{u}_3)\mu = a^{(3)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(3)}\partial_{xy}\mu + c^{(3)}\partial_x\mu + d^{(3)}\partial_y\mu + e^{(3)}\mu = f^{(3)} \quad (57)$$

$$\mathcal{L}(\boldsymbol{u}_4)\mu = a^{(4)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(4)}\partial_{xy}\mu + c^{(4)}\partial_x\mu + d^{(4)}\partial_y\mu + e^{(4)}\mu = f^{(4)}. \quad (58)$$

Equations (55)–(58) are ordered to ensure that $a^{(1)}b^{(2)} - b^{(1)}a^{(2)} \neq 0$. Then we solve (55) and (56) for the values $(\partial_{yy} - \partial_{xx})(\mu)$ and $\partial_{xy}\mu$. Substituting the result into (57) and (58) and simplifying leads us to

$$\mu_x + Q(x, y)\mu = 0 \quad (59)$$

$$\mu_y + R(x, y)\mu = 0. \quad (60)$$

A solvability condition on (59) and (60), which we assume to be satisfied since the given displacement fields are mutually compatible, is

$$\partial_y Q(x, y) = \partial_x R(x, y). \quad (61)$$

The joint solution of equations (59) and (60) is (54). □

Lemma 4. *The maximum number of linearly independent mutually compatible equilibrium strain fields that can exist in an isotropic incompressible elastic material is four. By linearly independent, we mean the strain components and certain of their derivatives must satisfy a constraint equation derived below.*

Proof. To the set of equations (55)–(58), add the fifth equation

$$\mathcal{L}(\boldsymbol{u}_5)\mu = a^{(5)}(\partial_{yy} - \partial_{xx})(\mu) + b^{(5)}\partial_{xy}\mu + c^{(5)}\partial_x\mu + d^{(5)}\partial_y\mu + e^{(5)}\mu = f^{(5)} \equiv 0. \tag{62}$$

We note that, for this case, we assume that the strain fields are in equilibrium, and so therefore $f^{(j)} = 0$, $j = 1, \ldots, 5$. Equations (55)–(58) and (62) together form a homogeneous linear system of equations for the unknowns, $((\partial_{yy} - \partial_{xx})\mu, \partial_{xy}\mu, \partial_x\mu, \partial_y\mu, \mu)^T$. Since, by assumption, the five strain fields are mutually compatible, we are guaranteed that a nontrivial solution exists. Therefore, the coefficient matrix formed by the five strain fields must be singular. Thus the strain fields are linearly dependent in this case. □

5. Conclusions

In this paper we have considered the inverse problem associated with elastic modulus imaging. We have shown that using a single displacement field to reconstruct the shear modulus requires a large amount of *a priori* knowledge related to the shear modulus itself. In medical imaging practice, the *a priori* information necessary is generally unobtainable. We have also shown, however, that using two different displacement fields reduces the number of undetermined coefficients in the total solution to at most four. To evaluate the unique solution, therefore, only four independent pieces of *a priori* knowledge regarding μ must be given. This, we believe, is perfectly reasonable in practice, as these values may be evaluated at or near the skin surface. Further, we have shown that in those circumstances in which four independent displacement fields can be measured, the modulus can be determined uniquely up to a multiplicative constant, with no *a priori* information. We must emphasize that all these conclusions were drawn for both quasistatic *and* transient deformations, and thus apply equally to both cases.

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