A simple absorbing boundary condition for FDTD modeling of lossy, dispersive media based on the one-way wave equation.

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Abstract— **Motivated by previous work on modeling dispersive media with a single pole, Z-transform conductivity model, we present an absorbing boundary condition (ABC) for such media, based on the one-way wave equation. The applicability of the method is tested on a 3D FDTD grid excited with a broadband Gaussian pulse, modulated at 2GHz. For such media and this high frequency range, the resulting reflection ratio for normal incidence is less than 1%. A comparison in 2D grids with the original perfectly matched layer (PML) ABC for dispersive media shows that for small angles of incidence, this Mur-type ABC is superior. In addition, it requires no additional storage of field components, is easy to implement, and is readily parallelizable. Therefore, despite its limitations, it can be a good alternative to other PML-based ABC's for lossy dispersive media, in high frequency applications such as underground radar or microwave imaging.**

Keywords— **Absorbing boundary conditions, dispersive media, FDTD, one-wave ABC**

I. Introduction

Absorbing boundary conditions (ABC) for the termination of FDTD lattices [1] are necessary in order to prevent nonphysical reflections from outgoing waves for a range of incident angles as wide as possible. The performance of the first ABC's that were developed [2], [3] deteriorated as the incidence angle diverged from normal, but the introduction of the perfectly matched layer (PML) [4] guaranteed good absorption for all angles. For many applications, the absorbing boundary condition needs to be modified in order to account for lossy dispersive media such as clay loam soil or biological tissue. Several methods to extend the PML in order to account for dispersive and/or lossy media media have been proposed, leading to high absorption [5]-[8]. One disadvantage of most of these formulations, such as the commonly used uniaxial PML (UPML) [5], is that they require at least two additional variables per electric and one per magnetic field component, increasing significantly the computational cost, especially for three dimensional cases. In this study, an absorbing boundary condition, based on the original Enquist-Majda ABCs [3], is developed and tested on a 3D grid. The results show that this ABC performs better in lossy than lossless dielectric and is superior to an eight layer original PML for small angles of incidence, while it leads to memory savings in comparison to the very effective uniaxial PML.

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II. First and second order absorbing boundary conditions

The right-going lossless one-way wave equation for electric or magnetic fields $W(t)$ at a given boundary x=0 is given in the time domain as [2],

$$\{\partial_x - c^{-1}[1-(cS_y)^2-(cS_z)^2]^{1/2}\partial_t\}W(t)_{x=0} = 0 \quad (1)$$

where c is the velocity of the medium and the S_y, S_z operators are defined as the inverse velocity components, such that $S_x^2+S_y^2+S_z^2 = 1/c^2$. This absorbing boundary condition can be written in the frequency domain as:

$$\{\partial_x - v^{-1}[1-(vS_y(\omega))^2-(vS_z(\omega))^2]^{1/2}j\omega\}W(\omega)_{x=0} = 0 \quad (2)$$

where $S_\gamma(\omega) = \partial_\gamma/(j\omega)$ is the frequency domain equivalent of the time domain operator defined in [2], and $v = (\varepsilon_c(\omega)\mu_0)^{-1/2}$ is the complex wave velocity for the medium, with $\varepsilon_c(\omega) = \varepsilon_0(\varepsilon^{'}(\omega) - \frac{j\sigma(\omega)}{\omega\varepsilon_0})$.

In the literature, the dispersion models used in the development of PML-type ABC for such media have mostly considered Debye or Lorentz-type dispersions [6],[7]. For a variety of dispersive media it is possible to separate the modeling for the permittivity and conductivity, and model dispersion using an average permittivity value $\varepsilon^{'}$ and frequency dependent conductivity [9], given by a Z-transform rational function having one pole [10],

$$\sigma(Z) = \frac{b_0 + b_1Z^{-1} + b_2Z^{-2}}{1+a_1Z^{-1}} \quad (3)$$

where $Z = e^{j\omega\triangle t}$ for time step $\triangle t$, and the parameters a_1, b_0, b_1, and b_2 are computed by fitting to measured data. Studies on frequency dependent moist soil data [11] indicate an excellent fit for velocity within 5% and loss factor within 20% from 30 MHz to 3 GHz (for which conductivity changes by more than one order of magnitude, and dielectric constant decreases by 30%). Recently, good fit with data for different types of dispersive biological tissue using this approach has also been reported [12]. The order of approximation for the square root in (2) yields the first or second order absorbing boundary condition. It is important to note that, although (2) accounts for any type of linear, isotropic medium, our interest and subsequent analysis focuses on lossy and dispersive media, with properties that can be modeled with the above conductivity profile at the high frequency range of 30 MHz to several GHz's.

The first-order approximation sets the square root equal to unity and thus (2) can be written as:

$$\{\partial_x - j\omega\sqrt{\mu_0\varepsilon_0\varepsilon'(\omega)}[1-\frac{j\sigma(\omega)}{\omega\varepsilon_0\varepsilon'(\omega)}]^{1/2}\}W(\omega)_{x=0} = 0 \quad (4)$$

In the most general case, $\varepsilon_c(\omega)$ will have both real and imaginary parts dependent on frequency. In previous work [13], we have used the approximation $\sqrt{1-x} \cong 1 - x/2$, $x \ll 1$, based on the assumption of small conductivity relative to the medium's permittivity, which is valid for many cases of soil or biological tissue over a wide frequency band [10],[11]. Later in this paper, we examine the effectiveness of this approximation for wave absorption in lossy media. Alternatively, we can set $\sqrt{\varepsilon'(\omega) - j\frac{\sigma(\omega)}{\omega\varepsilon_0}} = e(\omega) - j\frac{s(\omega)}{\omega}$, and calculate these quantities directly. Then, no assumption is necessary and the absorbing boundary condition is valid for any lossy, frequency-dispersive medium described by [10]. The practical disadvantage of this approach is that we need to find the constant average permittivity e and the parameters $\alpha_1, \beta_0, \beta_1, \beta_2$, replacing a_1, b_0, b_1, b_2 in (3), that match the values for $e(\omega)$ and $s(\omega)$. Eq.(4) becomes,

$$\{\partial_x - \frac{e(\omega)}{c_0}j\omega - \eta_0 s(\omega)\}W(\omega)_{x=0} = 0 \quad (5)$$

where $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ and $c_0 = (\varepsilon_0\mu_0)^{-1/2}$. If we convert Eq. (5) to the Z-domain ($j\omega \rightarrow \frac{1-Z^{-1}}{\triangle t}$) and use (3), and then convert it back to the time domain, we get

$$\begin{aligned}&\{(\partial_x - \tfrac{e}{c_0}\partial_t)(W^n + \alpha_1 W^{n-1}) - \eta_0(\beta_0 W^n \\ &+\beta_1 W^{n-1} + \beta_2 W^{n-2})\}_{x=0} = 0\end{aligned} \quad (6)$$

where $W^n = W(n\triangle t)$.

Using the second order approximation for the square root of the velocity terms in (2) leads to:

$$\{\partial_x - v^{-1}[1-\frac{1}{2}((cS_y)^2 - (cS_z)^2)]j\omega\}W(\omega)_{x=0} = 0 \quad (7)$$

Then following the same procedure as above, we derive the second order absorbing boundary condition,

$$\begin{aligned}&\{[u_1\partial_x + u_2 + u_3\beta_0 + \tfrac{1}{2}({\partial_y}^2 + {\partial_z}^2)]W^n \\ &+[u_1(\alpha_1 - 1)\partial_x + u_2(\alpha_1 - 2) + u_3(\beta_1 - \beta_0) \\ &+\tfrac{\alpha_1}{2}({\partial_y}^2 + {\partial_z}^2)]W^{n-1} \\ &+[-u_1\alpha_1\partial_x + u_2(1-2\alpha_1) + u_3(\beta_2 - \beta_1)]W^{n-2} \\ &+[u_2\alpha_1 - u_3\beta_2]W^{n-3}\}_{x=0} = 0\end{aligned} \quad (8)$$

where $u_1 = \frac{e}{c_0\triangle\Delta t}$, $u_2 = -\frac{e^2}{c_0^2\triangle t^2}$, $u_3 = -\frac{\mu_0}{2\triangle t}$.

III. Results and evaluation of the method

To analyze the efficiency of the method as well as some of its limitations, we have tested this one-wave ABC in two different numerical experiments. In the first experiment, the small conductivity approximation is used with non-dispersive media. Eq. (6) and (8) are used with $\alpha_1 = \beta_1 = \beta_2 = 0$, $\beta_0 = \frac{\sigma}{2\sqrt{\varepsilon'}}$, and $e = \sqrt{\varepsilon'}$. We excite

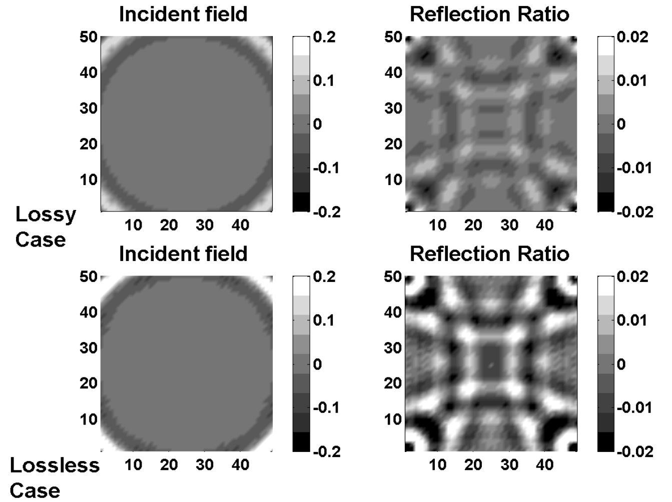

Fig. 1. Spatial distribution in the y-z plane of the electric field reflected from a 3D grid using the one-wave, lossy absorbing boundary condition in soil (top of the figure) and comparison with a dielectric case (bottom). As shown on the left, more than 80 % of the scaled incident field has reached the terminating boundaries at the time instant when the reflection ratio is calculated.

an E_x directed point source in the center of the grid with a wide-band, modulated Gaussian pulse of 2 GHz center frequency. The reflection percentage for all points in the y-z plane for the cases of non-dispersive, lossy and lossless dielectric (for comparison) is shown in Fig.1. The relative permittivity for these materials is 6.27 and the conductivity for the lossy material is 0.15 s/m. As shown in the figure, the resulting reflection ratio is less than 1%, except for the stronger corner effects. The stronger reflection in the lossless dielectric case shows that this ABC takes advantage of the lossy nature of the material. In fact, the performance improves as the conductivity rises until a certain value, after which the approximation does not hold. Indeed, a conductivity of 0.7 s/m for this permittivity value and frequency results to reflection of more than 15%. The above analysis indicates that when the small conductivity approximation approach is used in lossy, non-dispersive media, its performance depends upon the medium's loss tangent $\sigma/\omega\varepsilon'$.

In the second experiment, we consider dispersive Puerto-Rican clay loam of 10% moisture and 1.4 g/cm^3, for which σ varies from 0.008 to 0.15 s/m and ε' varies from 8.5 to 6.05, and apply the derived equations (6) and (8). For a time step $\Delta t = 20ps$, the propagation coefficients a_1, b_0, b_1, b_2 and ε_{av} are -0.9535, 1.076, -2.07289, 0.997138 and 3.679, and the ABC coefficients α_1, β_0, β_1, β_2 and e are -0.8785, 0.186521, -0.341021, 0.154628 and 2.002 respectively. For a plane wave excitation with a 2 GHz modulated Gaussian pulse in a 2D grid, the reflection ratio as a function of incidence angle is shown in Fig.2. The ABC is compared to the conventional and a modified eight-layer PML, tuned for this type of soil medium [9]. Fig.2 shows that this one-way wave ABC is superior for angles of incidence up to 25^o, while its performance deteriorates significantly for greater incidence angles, in the same fashion as

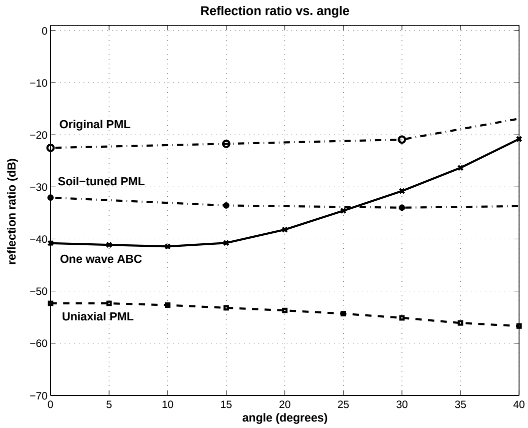

Fig. 2. Reflection ratio as a function of the incidence angle for the one-wave lossy ABC vs. various PML formulations. The "soil-tuned PML" is a modified version of the PML to account for the scattering medium conductivity and dispersion [9]. For small angles of incidence the one-wave ABC leads to superior performance relative to the original PML, and is comparable to a 5-layer UPML, which requires three additional variables per each field component.

the original Mur ABC fails as the incidence diverges from normal. A five-layer uniaxial PML (UPML) [7] for lossy inhomogeneous media, modified in a manner to account for this type of dispersion, was also implemented, leading to about 10 dB better performance for angles up to 20^o. However, this and other similar formulations, such as the recently presented convolutional PML (CPML)[8], require two additional variables per each electric and one per each magnetic field component, causing an increase in the computational burden, which for a lossy, non-dispersive case was calculated in [1]. The presence of three additional variables that store past field values for our dispersive model results to a further increase of additional memory equal to $48 \times N_{UPML} \times N^2 - 96 \times N_{UPML} \times N + 1152 \times N_{UPML}^2$ for a 3D cubic lattice relative to the $15 \times N^3$ total field variables. Then for a $100 \times 100 \times 100$ problem and a five-layer UPML (or CPML), this amounts to 16% of additional memory requirements and this percentage increases as N decreases. This increase in the computational cost prompts us to believe that the absorbing boundary condition presented here may have a place in general FDTD computations for lossy and/or dispersive media.

IV. Conclusions

An ABC based on Mur's approach has been developed for dispersive media modeled with a single pole conductivity Z-transform model. This ABC does not require additional storage of field components or extra ABC layers, as would be necessary with the PML, and is simple to implement for 3D geometries. Two possible implementations were discussed; the performance of the first, based on a small conductivity assumption, depends upon the medium's loss tangent, while the second is more general and accurate but requires an additional tuning of the square root of the medium's complex permittivity. Despite these limitations and although higher absorption can be achieved using modified PML techniques at an increased computational cost, we believe that this Mur-type ABC is a good alternative for dispersive media, when modeled with the single pole conductivity model which was presented here and in previous work.

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